Dynamical phase transitions and self-organized criticality in a theoretical spring-block model

W.-S. Liu* and Y. N. Lu
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China

E. J. Ding
Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing, 100080, China;
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China;
and Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China
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We numerically and analytically study the self-organized criticality of a nonconservative and globally coupled spring-block model. With a spring parameter and a time-scale parameter, this model displays the dynamical states of two types which are different from each other in three aspects: the stick-slip processes, the strain distribution functions, and the power laws of shock-size distributions. Over a range of parameter space these two states may coexist. The transitions between these two states as well as the phenomenon of hysteresis are first observed in the context of spring-block models. A phase diagram is roughly sketched.

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I. INTRODUCTION

A variety of complex systems in nature demonstrate scaling properties with power-law correlations. The general examples include complex spatial patterns and structures such as coastlines [1], and irreversible growth structures [2], and diverse temporal processes as in sand flow [3,4], driven diffusive systems [5,6], and traffic transportation [7]. These phenomena lack natural length and time scales and instead possess scale-invariant or self-similar features. The geometrical aspects of a scale-invariant system have been successfully characterized by means of *fractals* [1], while the necessary analytical tools may be expected from the studies of critical phenomena in statistical physics. In static critical phenomena, the system exhibits fluctuations in all possible length scales with the scale invariance, but the phase transition occurs at a critical point reached only by fine tuning some parameters of the system. By contrast, for the dynamic phenomena with self-similarity where no fine tuning is needed, Bak, Tang, and Wiesenfeld (BTW) argued and demonstrated that dynamic systems naturally evolve into self-organized critical states robust with respect to variations of parameters and initial conditions [4]. This generic behavior has been called self-organized criticality (SOC) which is suggested probably to be the common underlying mechanism behind the phenomena described above.

The term SOC applies to systems that are driven either completely deterministically at a very low rate, or by random local perturbations, i.e., slow noise. A fundamental result for driven nonequilibrium systems subject to external white noise is that scale invariance can indeed occur generically, but only in systems with either a conservation law or a specially continuous symmetry, such as the translational invariance which allows even equilibrium interfaces to exhibit rough, scale-invariant phases [5,8]. For these noise-driven systems it is believed that the necessary requirement for SOC is that the dynamics satisfy a conservation law. Nevertheless, with regard to the generality of the concept of SOC, this is a critical aspect, since many natural phenomena have inherent nonconservative properties. The discovery of the deterministic cellular-automaton model [9] displaying SOC without a conservation law is of course somewhat surprising and very interesting, since it suggests a different mechanism for the generic generation of scale-invariant structures. Then a few nonconservative dynamic models have recently been argued to exhibit SOC [10,11].

In nature, earthquakes are probably the most relevant paradigm of SOC, since there are several kinds of scale invariance in the earthquake processes, which are given in a power-law form in several empirical formulas (see, e.g., [12–15]). For example, the distribution of energy released during earthquakes has been found to obey the famous Gutenberg-Richter law [12]. The law is based on the empirical observation that the number N of earthquakes of magnitude M greater than m is given by the relation

$$\log_{10} N(M > m) = a - bm, \tag{1}$$

where the precise values of a and b depend on the location, but empirically b is in the interval $0.8 < b < 1.5$ [16]. The energy E (or other physical quantities such as the "seismic moment") released during the earthquake is believed to increase exponentially with the earthquake magnitude,

*Present address: Department of Physics, The University of Texas at Austin, Austin, TX 78712.

$$\log_{10} E = c + dm, \tag{2}$$

where the value of d is 1 and $\frac{3}{2}$ for small and large earthquakes, respectively [17]. Thus the Gutenberg-Richter law is essentially a power law connecting the frequency distribution function with energy greater than E,

$$N(E) \sim E^{-B}, \tag{3}$$

with $0.8 < B < 1.05$. Immediately after the introduction of the idea of SOC by Bak, Tang, and Wiesenfeld, it was suggested that SOC might be a good explanation for the observed power laws in earthquakes [18–20]. However, most of these suggested models are conservative and have no physical interpretation in the context of earthquakes.

The properties of extended fractures in earthquakes have been explored in a number of earthquake model papers, dynamically [10,21–24] and statistically [25]. One model that has received considerable attention is the stick-slip model of Burridge and Knopoff (BK) [21] where chains of metal blocks connected by springs are dragged along a surface. This model demonstrates that the number of shocks with energy release above E follows Eq. (3) with $B \simeq 1$ in their one-dimensional system. Related to the scaling properties in earthquake processes, dynamic phase transitions have been observed. Takayasu and Matsuzaki [26] extended the one-dimensional Burridge-Knopoff model to a multidimensional mechanical model and found that it has an order-disorder phase transition, above which the entire system slips together as one cluster. The behavior of the system depends on the strength of the coupling. In the critical state of this model, the size distribution of clusters follows the same power law as that of earthquakes. Their two-dimensional model is similar to the BTW model. Recently, with a nonconservative self-organized critical model equivalent to a quasistatic two-dimensional version of the spring-block model of Burridge-Knopoff, the nonuniversality of the critical exponent and a dynamical phase transition from localized to nonlocalized behavior are found as the level of conservation is increased [10]. As in a version considered by Carlson and Langer [22], nonlinear friction with respect to the velocity is introduced to a homogeneous BK spring-block model so that the parameters of it divide the behavior of the system into qualitatively distinct regimes [24]. By varying one of the parameters, this model appears to exhibit a continuous transition from a state where all the blocks are continuously moving to another one where the motion occurs in short, but violent, ruptures.

Comparing with the earthquake models, the statistical properties of the ruptures of fiber bundles are a classical problem which has been studied over a number of years apparently initiated by the work of Peirce [27]. Failure is often associated with the statistics of extremes [28–30] and thus is typically studied through the model of links with randomly distributed failure thresholds associated in series [30]. Daniels first considered the following fiber-bundle problem: parallel vertical lines with identical spring constant but random failure thresholds where the total stress is equally shared among the links [31]. Note that the problem is posed in similar terms in the electrical or mechanical context with the correspondence of spring constant to link conductance, mechanical stress to electric current density, and mechanical strain to electric voltage [32]. It can model a variety of systems, such as cables or ropes made of numerous fibers, geological faults which are locked by asperity barriers sharing the total stress [33,34], electric networks, etc. Also, such a fiber bundle has been discussed further recently [35,36], and been generalized by connecting bundles of parallel fibers in series [37,38]. Fiber-bundle models such as these, which have at least a superficial resemblance with the earthquake problem, demonstrate the existence of the power-law distribution of failure.

Motivated by both BK and fiber-bundle models, we introduce in this paper a nonconservative, globally coupled, spring-block model, which is the simplified variant of the BK model by mimicking the extended fractures in earthquakes and yet is referred to bundles of parallel fibers in series. Since the presence of different time scales is supposed to characterize some properties of failure events, as discussed by Gabrielov, Newman, and Knopoff through quasistatic lattice models [39], a time-scale parameter has been defined in this model so that the dynamical process is considered more carefully than in some previous ones [10,40]. Numerical and analytical studies show that this model differs greatly from the previous ones, with fascinating features, i.e., the coexistence of two distinct states and the phenomenon of hysteresis in the transition between these two states, which was reported briefly in Ref. [41].

In Sec. II, the model is described in detail. Section III serves as the illustration of two different types of dynamical states in this system. The phase transitions between the two states are discussed in Sec. IV by simulations.

II. MODEL

A. General description

Suppose that a total number N of blocks are placed on a carpet (see Fig. 1). All blocks are connected to a rigid bar individually through a total number N of springs which are of the same elastic modulus κ_2. In addition, the bar is connected to a trunk spring with elastic modulus κ_1, and the other tip of the trunk spring, A, is drawn steadily. We assume that there is only static friction between every block and the carpet, and that the carpet is nonuniform so that the maximum static friction may be different at various places on it. Introducing a control parameter $\kappa = \kappa_1/(\kappa_2 N)$ [40], we obtain the equilibrium condition

$$N\kappa(d - z) = \sum_{i=1}^{N} x_i \tag{4}$$

for the rigid bar, where x_i is the strain of the ith spring, and d and z are the displacements of the tip A and the bar, respectively. The threshold of strain, t_i, for the ith spring, corresponding to the maximum static friction between the ith block and the carpet, is picked from a prob-

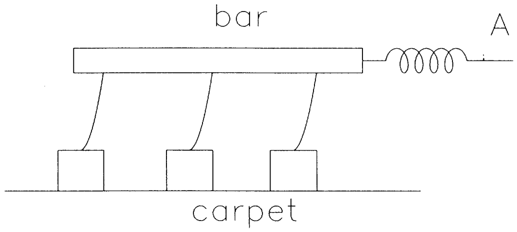

FIG. 1. A sketch of the spring-block system under consideration.

ability $p_{th}(t)$, called the threshold distribution function [30,23]. Beginning from some initial conditions, d, z, and x_i $(1 \leq i \leq N)$ increase uniformly as the tip A is drawn steadily. When the strain of some spring, say the ith one, x_i, reaches its threshold t_i, the block will move. Letting $y_i = t_i - x_i$, we note that the block with the smallest y_i must slip first. Since the spring connected to the slipping block no longer carries load, the trunk spring will relax a little, whereas the loads of the other springs will increase. Therefore the strains of other springs may exceed their thresholds, and the corresponding blocks slip too. So a chain reaction, or a shock, might occur in the spring-block system.

B. The presence of three time scales

In the spring-block models studied previously it was usually assumed that the slipping block could restick to the carpet at once [10,40]. However, in a real spring-block system the loads on the springs are redistributed among the springs as the block is moving. For this model, we have three different scales of time: the time τ_A needed for the tip A to move a unit distance, the relaxation time τ_R for the redistribution of the load among the springs, and the time τ_S for the moving block to restick to the carpet. Since the time interval between earthquakes is much larger than the actual duration of an earthquake, we assume that $\tau_S \ll \tau_A$ and $\tau_R \ll \tau_A$, yet the ratio of τ_S to τ_R remains variable. For the case of $\tau_S \ll \tau_R$, as considered previously [10,40], the moving block could restick to a new place long before the springs rearrange their strain distributions. If $\tau_S \gg \tau_R$, however, any moving block cannot restick until the chain reaction ceases. Taking the interval between two shocks as the time unit, we have that the total number T of the shocks is simply the total time of the process.

In order to discuss the general case, we introduce a time-scale parameter [39] $\lambda \in [0,1]$. This parameter is expected to describe roughly the ratio $\tau_S/(\tau_R N)$. The precise definition for this parameter is such that there could at most be λN springs carrying no load at the same time. Suppose that there have been a total number $m-1$ of blocks slipping at one time. When there is still some spring, say the jth one, the strain x_j of which is larger than the threshold t_j, the jth block will slip. If $m < \lambda N$, there will be a total number m of blocks slipping, and the load has to be distributed on the other $N-m$ springs. If $m = \lambda N$, the earliest slipping one within these λN blocks will restick on the carpet immediately after the jth block slips. Therefore, owing to the slip of the jth block, the strains of the springs corresponding to the sticking blocks will increase δx,

$$\delta x = \begin{cases} \frac{x_j}{(\kappa+1)N-m} & \text{if } m < \lambda N, \\ \frac{x_j}{(\kappa-\lambda+1)N} & \text{if } m = \lambda N. \end{cases} \quad (5)$$

It is clear that $\lambda = 0$ corresponds to the limiting case $\tau_S \ll \tau_R$ previously studied in detail in Ref. [40], and $\lambda = 1$ to $\tau_S \gg \tau_R$. In the present paper, we generally discuss the case of $\lambda > 0$. It is worth pointing out at last that we do not actually consider explicitly the different fast time scales but substitute the ratio of them by a restriction on the maximum possible number of simultaneously slipping blocks. However, this leaves open the question of the correspondence of the effects of this restriction and the possible effects of actual fast time scales [39].

III. THE TWO DYNAMICAL STATES

Let us first investigate a limiting case $\lambda = 1$. In simulations we take a uniform threshold distribution $p_{th}(t) = 1$ for $0 \leq t \leq 1$, and choose $\kappa = 0.25$ and $N = 10\,000$. Depending on the initial conditions the system may evolve into two types of states. For example, if the initial distribution of $\{y_i\}$ is $p_0(y) = 2(1-y)$ for $0 \leq y \leq 1$, the state into which the system evolves is called type I. If, on the other hand, the initial distribution is $p_0(y) = 1$ for $0 \leq y \leq 1$, the state in which the system finally dwells is called type II. For any initial conditions the system evolves always into one of these two states with a probability. There are three pieces of evidence, i.e., the stick-slip processes, the strain distribution functions, and the power laws of shock-size distributions, convincingly showing that these two types of states are different from each other.

A. The stick-slip process of blocks

The stick-slip processes of the two types are qualitatively different. Figure 2 shows the typical cases of the stick-slip processes of two types. The total number Δ of blocks which slipped during a shock is defined as the size of the shock. A series of large-size shocks with a size $O(N)$ occur almost periodically in type II, where almost all blocks slip simultaneously so one may call it "superslip" event. Such a "superslip" event has also been observed in other spring-block models [24,26]. By contrast, in type I, there are normally small-size $(\Delta \ll N)$ shocks distributed randomly with respect to time. Nevertheless, we also consider the system of a small number of blocks far from the initial strain distribution in order to robustly distinguish chaotic behavior of the model from complicated periodical behavior in type II. Simulations with $N = 300$ after 2×10^5 transient processes confirm the periodical behavior in all the times as long as the system is kept in type II, given in Fig. 3 as T from 0 to $10\,000$.

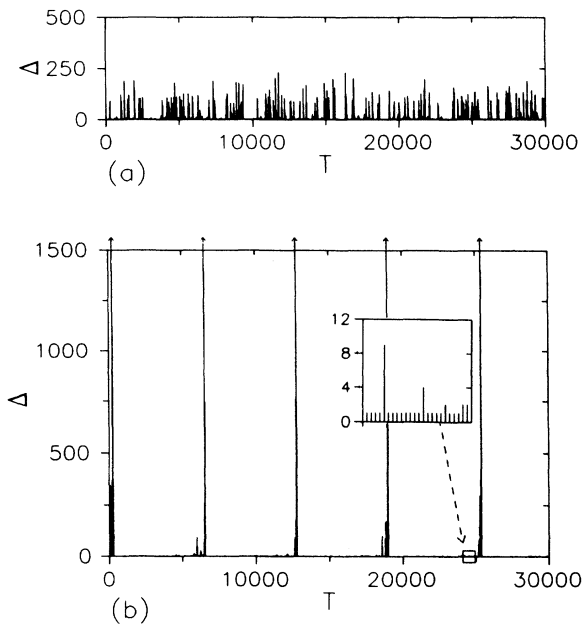

FIG. 2. The typical stick-slip processes for the states of two types. They are obtained with $N = 10\,000$, $\kappa = 0.25$, and $\lambda = 1$. The configurations of two types are different due to the initial states of the system. (a) type I; (b) type II. The lines with arrows away from the tops of the columns in (b) indicate that $\Delta \simeq 10\,000$.

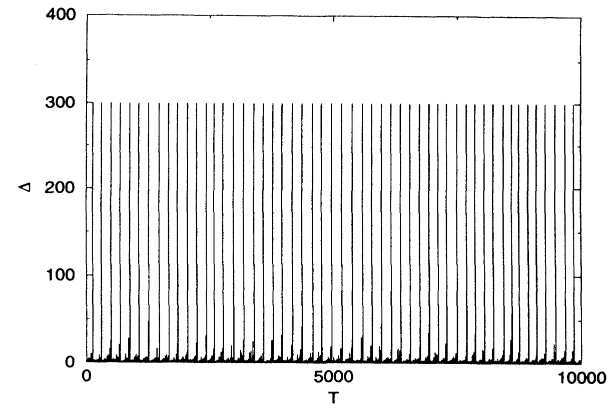

FIG. 3. The complicated periodical behavior of a small number $N = 300$ of blocks with $\lambda = 1$ and $\kappa = 0.25$. The relaxation time is 2×10^5 while the system is kept in type II.

B. The evolutionary functions of strain distributions

The evolutions of the strain distribution of $\{y_i\}$ in these two types of states are different also. It can be demonstrated that the strain distribution function is stationary for type I but periodically changeable for type II. In order to see this conclusion, let us consider the shocks with small size $\Delta \ll N$. Assume that $\{y_i\}$ is the *ordered* sequence $y_1 \le y_2 \le \cdots \le y_N$. In a shock of size Δ starting at a certain place where the displacement of the bar is z, we know from Eq. (5) that after a number j of blocks slip, the bar will move forward by a distance

$$\delta_j = \frac{\sum_{i=1}^{j} x_i}{(\kappa+1)N - j}, \quad j = 1, 2, \ldots, \Delta. \tag{6}$$

Since the displacement z of the bar monotonically increases with the time T, z may take the role of time. Denoting by $p(y,z)$ the distribution of $\{y_i\}$ at "time" z, we have for any shock of size Δ that

$$p(y, z + \delta_\Delta) = p(y + \delta_\Delta, z) + p_{th}(y) \int_0^{\delta_\Delta} p(\xi, z)\, d\xi,$$

$$y \in [0, 1 - \delta_\Delta], \quad z > 0. \tag{7}$$

For type I, as small Δ makes $\delta_\Delta \ll 1$, Eq. (7) may be written as a partial differential equation,

$$\frac{\partial p(y,z)}{\partial z} = \frac{\partial p(y,z)}{\partial y} + p_{th}(y) p(0,z),$$

$$y \in [0,1], \; z > 0,$$

$$p(y,0) = p_0(y), \quad y \in [0,1], \qquad p(1,z) = 0, \quad z > 0, \tag{8}$$

where $p_0(y)$ is the initial distribution of y_i's.

For a uniform distribution $p_{th}(t) = 1$ in $[0,1]$, the solution of Eq. (8) is easily given as

$$p(y,z) = \begin{cases} p_0(y+z) + e^z \int_0^z p_0(\xi) e^{-\xi}\, d\xi, & y \le 1 - z, \\ e^z \int_0^z p_0(\xi) e^{-\xi}\, d\xi - e^{z+y-1} \int_0^{z+y-1} p_0(\xi) e^{-\xi}\, d\xi, & 1 - z < y \le 1, \end{cases} \tag{9}$$

when $z \le 1$. For $n < z \le n+1$ the variable z in Eq. (9) should be replaced by $z - n$ and the function $p_0(y)$ by $p_n(y)$, where

$$p_n(y) \equiv p(y,n) = e \int_0^1 p_{n-1}(\xi) e^{-\xi}\, d\xi - e^y \int_0^y p_{n-1}(\xi) e^{-\xi}\, d\xi. \tag{10}$$

As $z \to \infty$ the distribution $p(y,z)$ evolves into a stationary state, $p_\infty(y) = 2(1-y)$, independent of the initial distribution $p_0(y)$. This is confirmed by our simulation. In general, we know from Eq. (8) that, for an arbitrary distribution $p_{th}(t)$ of $\{t_i\}$, the stationary solution $p_\infty(y)$ of $p(y,z)$ must satisfy $\partial p(y,z)/\partial z = 0$. Hence,

$$\frac{dp_\infty(y)}{dy} + p_{th}(y)\, p_\infty(0) = 0. \tag{11}$$

Then we have that

$$p_\infty(y) = N_p \int_y^1 p_{th}(\xi)\, d\xi, \tag{12}$$

where the coefficient N_p stands for the normalization constant. The stationary solution $p_\infty(y)$ can be reached by the system when $z \to \infty$.

We can also transform Eq. (7) to the integral equation

$$p(y,z) = \begin{cases} p_0(z+y) + \int_0^z d\xi\; p_{th}(\xi+y)p(0,z-\xi), & y \le 1-z,\ z \le 1, \\ \int_0^{1-y} d\xi\; p_{th}(\xi+y)p(0,z-\xi), & 1-z < y \le 1, z \le 1, \\ \int_0^{1-y} d\xi\; p_{th}(\xi+y)p(0,z-\xi), & z > 1. \end{cases} \tag{13}$$

Let $q(x,z)$ denote the distribution of x_i's at "time" z. Since $t = x+y$ has the distribution $p_{th}(x+y)$ and y has $p(y,z)$, we can express $q(x,z)$ with the formula

$$q(x,z) = \int_0^{1-x} dy\; p(y,z)\gamma(x,y,z),$$

where $\gamma(x,y,z)$ is the probability condition of strain x with both y and z given, and can be expressed as

$$\gamma(x,y,z) = \begin{cases} \frac{p_{th}(x+y)[p_0(y+z)+p(0,z-x)]}{p(y,z)}, & y \le 1-z, \\ \frac{p_{th}(x+y)p(0,z-x)}{p(y,z)}, & y > 1-z, z \le 1, \\ \frac{p_{th}(x+y)p(0,z-x)}{p(y,z)}, & z > 1. \end{cases} \tag{14}$$

Obviously, only if z may diverge in type I while the above equations remain applicable may $\gamma(x,0,z)$ reach a stationary solution together with $p(y,z)$, i.e.,

$$\gamma(x,0,z\to\infty) = p_{th}(x). \tag{15}$$

For type II, since the system is of periodical shocks of very large size ($\Delta = N$) as show in Fig. 2(b), the integral Eq. (13) is applicable only between two large-size shocks. We argue that the distribution function $p(y,z)$ would not be stationary but rather periodically changed with respect to z. Let us consider that the system produces a violent shock of size $\Delta = N$ at "time" $z = Z_1$. Immediately after this moment, the strains x_i of all springs are zero, so that $p(y, Z_1+0) = p_{th}(y)$. Then $p(y,z)$ will evolve repeatedly with z according to the integral Eq. (13) until the next global slip takes place at some "time" $Z_2 = Z_1 + z_p$ (z_p will be analytically given somewhat later) where $p(y,Z_2) = p(y,Z_1) = p_{th}(y)$. Therefore one has a nearly periodical function $p(y,z+z_p) = p(y,z)$, and correspondingly a periodical $\gamma(x,y,z)$ with respect to z for this type as well. Simulations confirm that the distribution $p(y,z)$ for type II is indeed a periodically evolutionary function, and that immediately after every global slip it is the same in statistics.

C. The different shock-size distributions for two states

Now we are going to determine the shock-size distributions $D_I(\Delta)$ and $D_{II}(\Delta)$ for states of type I and II, respectively. The algorithm is mainly based on that in Ref. [36]. As described before, the $\{y_i\}$'s appear to be in the *ordered* sequence $y_1 \le y_2 \le \cdots \le y_N$. Denoting by d the displacement of the tip A, we see from Eq. (4) that in order to move the first block the tip A should reach $d_1 = d_0 + y_1(\kappa+1)/\kappa$ apart from d_0, the initial position of A. In general, in order to move the jth block the displacement of A should be

$$d_j = d_0 + \frac{N(\kappa+1)+1-j}{N\kappa} y_j - \frac{1}{N\kappa}\sum_{i=1}^{j-1} x_i. \tag{16}$$

For a shock of size Δ, a forward condition and a backward condition must be satisfied.

1. The forward condition

The forward condition mentioned is

$$d_j < d_1, \qquad j = 2,3,\ldots,\Delta, \tag{17}$$

and

$$d_{\Delta+1} > d_1. \tag{18}$$

The two equations (17) and (18) are, respectively, equivalent to

$$y_j < \frac{N(\kappa+1)}{N(\kappa+1)+1-j} y_1 + \frac{1}{N(\kappa+1)+1-j}\sum_{i=1}^{j-1} x_i, \qquad j = 2,3,\ldots,\Delta, \tag{19}$$

and

$$y_{\Delta+1} > \frac{N(\kappa+1)}{N(\kappa+1)-\Delta} y_1 + \frac{1}{N(\kappa+1)-\Delta}\sum_{i=1}^{\Delta} x_i. \tag{20}$$

Assuming $\Delta \ll N$ and letting $\alpha = 1/(\kappa+1)$, one may simplify Eqs. (19) and (20) to

$$y_j < y_1 + \delta_{j-1}, \qquad j = 2, 3, \ldots, \Delta, \tag{21}$$

and

$$y_{\Delta+1} > y_1 + \delta_\Delta, \tag{22}$$

with

$$\delta_j = \frac{1}{N(\kappa+1) - j} \sum_{i=1}^{j} x_i \simeq \frac{\alpha}{N} \sum_{i=1}^{j} x_i. \tag{23}$$

The simplified forward condition (21) and (22) means that the $\Delta - 1$ y values $y_2, \ldots, y_\Delta$ must lie in the interval $(y_1, y_1 + \delta_{\Delta-1})$, and $N - \Delta$ values $y_{\Delta+1}, y_{\Delta+2}, \ldots, y_N$ must be larger than $y_1 + \delta_\Delta$. The probability for this is

$$\psi^*(\Delta, x_i, z) = \binom{N-1}{\Delta-1} \times \left[\frac{P(y_1 + \delta_{\Delta-1}, z) - P(y_1, z)}{1 - P(y_1, z)}\right]^{\Delta-1} \times \left[\frac{1 - P(y_1 + \delta_\Delta, z)}{1 - P(y_1, z)}\right]^{N-\Delta}, \tag{24}$$

where $P(y,z) \equiv \int_0^y p(\xi, z) d\xi$. Since $\Delta \ll N$ is assumed, we may expand $P(y,z)$ in δ_Δ to the first order, thus obtaining

$$\psi^*(\Delta, x_i, z) = \binom{N-1}{\Delta-1} \left[\frac{p(y_1, z)\delta_{\Delta-1}}{1 - P(y_1, z)}\right]^{\Delta-1} \times \left[1 - \frac{p(y_1, z)\delta_\Delta}{1 - P(y_1, z)}\right]^{N-\Delta}. \tag{25}$$

The last factor on the right-hand side of this expression is, for large $N - \Delta$, essentially

$$\exp\left[-\frac{\alpha p(y_1, z)}{1 - P(y_1, z)} \sum_{i=1}^{\Delta} x_i\right], \tag{26}$$

where Eq. (23) has been used. The binomial coefficient in Eq. (25) may for $N \gg \Delta$ be approximated by

$$\binom{N-1}{\Delta-1} \simeq \frac{(N-1)^{\Delta-1}}{(\Delta-1)!}. \tag{27}$$

Noticing $y_1 \simeq 0$ and denoting $s(z) = \alpha p(0, z)$, one derives from Eq. (25) that

$$\psi^*(\Delta, x_i, z) = \frac{1}{(\Delta-1)!} \left[s(z) \sum_{i=1}^{\Delta-1} x_i\right]^{\Delta-1} \times \exp\left[-s(z) \sum_{i=1}^{\Delta} x_i\right]. \tag{28}$$

Integrating the above equation over x_i $(i = 1, 2, \ldots, \Delta - 1)$ we get the probability

$$\Psi^*(\Delta, z) = \frac{s(z)^{\Delta-1}}{(\Delta-1)!} I_0(s(z)) \left[\prod_{i=1}^{\Delta-1} \int_0^1 dx_i \gamma(x_i, 0, z)\right] \times \left[\sum_{i=1}^{\Delta-1} x_i\right]^{\Delta-1} \exp\left[-s(z) \sum_{i=1}^{\Delta-1} x_i\right], \tag{29}$$

with $I_0(s(z)) = \int_0^1 e^{-s(z)x} \gamma(x, 0, z)\, dx$.

It remains to secure that all the $\Delta - 1$ inequalities (21) are fulfilled, i.e., that $y_2 < y_1 + \delta_1, y_3 < y_1 + \delta_2$, etc., given that all the $\Delta - 1$ elements lie in the interval $(y_1, y_1 + \delta_{\Delta-1})$. Since the probability density can be considered constant in the small interval, this is equivalent to the following combinatorial problem: $\Delta - 1$ elements are to be randomly distributed among $\Delta - 1$ numbered slots, and the first slot contains at least one element, the two first slots contains at least two elements, and so on. This additional condition can be described by

$$\sum_{i=1}^{J} f_i \geq J, \qquad J = 1, 2, \ldots, \Delta - 2, \tag{30}$$

with $\sum_{i=1}^{\Delta-1} f_i = \Delta - 1$, where f_i is the number of elements in the ith slot. With these additional conditions considered, the forward probability is finally derived from Eq. (29) as

$$\Psi(\Delta, z) = \frac{s(z)^{\Delta-1}}{(\Delta-1)!} I_0(s(z)) \left[\prod_{i=1}^{\Delta-1} \int_0^1 dx_i \gamma(x_i, 0, z)\right] \times \exp\left[-s(z) \sum_{i=1}^{\Delta-1} x_i\right] \hat{S}_f \left[\sum_{i=1}^{\Delta-1} x_i\right]^{\Delta-1}, \tag{31}$$

where $\hat{S}_f$ is an operator which selects those terms in the polynomial satisfying the conditions (30).

2. Backward condition

It is obvious that shocks observed in experiments only occur at some values of z. The forward condition is, hence, not certainly sufficient in order to determine the distribution of the shocks with size Δ. Going through the whole experiment, we record the displacement z_i of the bar corresponding to the ith event of block slip when tip A is steadily driven. Since the block slips only if the corresponding $y = 0$, there is a distribution $p(0, z)$ of z_i's. Also assume z_i's in an ordered sequence, i.e., $z_1 \leq z_2 \leq \cdots \leq z_w$. The block-slip events in a whole process are $w = N \int_0^{z_w} dz\, p(0, z)$ as z increases from 0 to z_w. Assuming that in the beginning of the experiment $z = 0$ and correspondingly $d = \tilde{d}_0$, we find that in order to move the kth block the displacement of A must reach

$$d_k = \tilde{d}_0 + \frac{N(\kappa+1) + 1 - k}{N\kappa} z_k - \frac{1}{N\kappa} \sum_{i=1}^{k-1} x_i', \tag{32}$$

similar to Eq. (16), expecting that the x_i''s are the strains when $z = 0$. For a shock to start at $z = z_k$, we must secure that d_k exceeds all previous values,

$$d_{k-j} < d_k, \qquad j = 1, 2, \ldots, k-1, \tag{33}$$

which may be called the backward condition. Otherwise the shock may occur inside a large one, and consequently will not be recorded in experiments with driving of the tip A. For a large N, experiencing a procedure as done on the forward condition, we may equivalently transfer Eq. (33) into

$$z_{k-j} < z_k - \tilde{\delta}_j, \qquad j = 1, 2, \ldots, k-1, \tag{34}$$

where

$$\tilde{\delta}_j \simeq \frac{\alpha}{N} \sum_{i=1}^{j} x'_{k-i}. \tag{35}$$

It is clear that in general it is the first few neighboring values, $d_{k-1}, d_{k-2}, \ldots$, that are most likely to exceed d_k due to fluctuations. Let us therefore determine $\chi(L, z_k)$, the probability that none of the values $d_{k-1}, d_{k-2}, \ldots, d_{k-L}$ exceeds d_k, for $L \ll N$.

To determine $\chi(L, z_k)$ we need according to Eq. (34) the probability that

$$z_{k-j} < z_k - \tilde{\delta}_j, \qquad j = 1, 2, \ldots, L. \tag{36}$$

This implies that a number h, not exceeding $L-1$, of elements may lie in the interval $(z_k - \tilde{\delta}_L, z_k - \tilde{\delta}_1)$, while all the $k-1-h$ remaining elements must be smaller than $z_k - \tilde{\delta}_L$. The probability for this is

$$\binom{k-1}{h} \left[\frac{\alpha p(0, z_k)}{N Q(0, z_k)} \sum_{i=2}^{L} x'_{k-i} \right]^h \times \left[1 - \frac{\alpha p(0, z_k)}{N Q(0, z_k)} \sum_{i=1}^{L} x'_{k-i} \right]^{k-1-h}, \tag{37}$$

where $Q(y, z) = \int_0^z p(y, \xi)\, d\xi$ and $k = NQ(0, z_k)$. Since k is of order N and $N \gg 1$, the last factor on the right-hand side is essentially

$$\exp\left[-\frac{\alpha p(0, z_k) k}{N Q(0, z_k)} \sum_{i=1}^{L} x'_{k-i} \right] \tag{38}$$

and the factorial may for $k \gg h$ be approximated by

$$\binom{k-1}{h} \simeq \frac{k^h}{h!}. \tag{39}$$

With the notation of $NQ(0, z_k) = k$, we sum Eq. (37) over the allowed values of h and then have from Eq. (37) the probability

$$\sum_{h=0}^{L-1} \frac{s(z_k)^h}{h!} \exp\left[-s(z_k) \sum_{i=1}^{L} x'_{k-i} \right] \left[\sum_{i=2}^{L} x'_{k-i} \right]^h, \tag{40}$$

recalling $s(z) = \alpha p(0, z)$. Integrating the above equation over all x's we get the probability

$$\chi^*(L, z_k) = I_0(s(z_k)) \left[\prod_{i=2}^{L} \int_0^1 dx'_{k-i} \gamma(x'_{k-i}, 0, z_k) \right] \times \sum_{h=0}^{L-1} \frac{s(z_k)^h}{h!} \exp\left[-s(z_k) \sum_{i=2}^{L} x'_{k-i} \right] \times \left[\sum_{i=2}^{L} x'_{k-i} \right]^h. \tag{41}$$

It remains to secure that the h values in the interval $(z_k - \tilde{\delta}_L, z_k - \tilde{\delta}_1)$ fulfill Eq. (36). This is again a combinatorial problem with $L-1$ slots, such that the first slot to the right can at most contain one element, the two rightmost slots can at most contain two elements, and so on. This condition can be described by

$$\sum_{i=1}^{J} b_i \leq J, \qquad J = 1, 2, \ldots, L-1, \tag{42}$$

with $\sum_{i=1}^{L-1} b_i = h$, where b_i is the number of elements in the ith slot to the right. Equation (29) is thus modified to

$$\chi(L, z_k) = I_0(s(z_k)) \left[\prod_{i=2}^{L} \int_0^1 dx'_{k-i} \gamma(x'_{k-i}, 0, z_k) \right] \times \sum_{h=0}^{L-1} \frac{s(z_k)^h}{h!} \exp\left[-s(z_k) \sum_{i=2}^{L} x'_{k-i} \right] \times \hat{S}_b \left[\sum_{i=2}^{L} x'_{k-i} \right]^h, \tag{43}$$

where $\hat{S}_b$ is an operator which selects those terms in the polynomial satisfying the conditions (42). Finally, the backward probability for a shock starting at z under conditions (33) should be obtained by increasing $L \to \infty$ in the probability $\chi(L, z)$, i.e.,

$$m(z) \equiv \lim_{L \to \infty} \chi(L, z). \tag{44}$$

Let us roughly discuss the backward condition in another way which might give some physical insight. The probability for all conditions (42) to be fulfilled is (see the Appendix of Ref. [36])

$$\epsilon_{h, L-1} = (L - h) \frac{L^{h-1}}{(L-1)^h}. \tag{45}$$

Inserting the factor $\epsilon_{h,L-1}$ into Eq. (40), we have the desired backward probability

$$\chi(L,z_k)=\sum_{h=0}^{L-1}\epsilon_{h,L-1}\frac{s(z_k)^h}{h!}\exp\left[-s(z_k)\sum_{i=1}^{L}x'_{k-i}\right]\left[\sum_{i=2}^{L}x'_{k-i}\right]^h. \tag{46}$$

Since $N \gg L \gg 1$, we approximately have that $x'_{k-i}=t_{k-i}-y_{k-i}\simeq t_{k-i}$ $(i\le L)$ and therefore $\sum_{i=1}^{L}x'_{k-i}=L\langle t\rangle$ and $\sum_{i=2}^{L}x'_{k-i}=(L-1)\langle t\rangle$, where $\langle t\rangle=\int_0^1 t\,p_{th}(t)\,dt$. Then we obtain from Eq. (46) that

$$\chi(L,z_k)=\exp[-\langle t\rangle s(z_k)L]\sum_{h=0}^{L-1}\frac{L-h}{h!L}[\langle t\rangle s(z_k)L]^h \tag{47}$$

$$=[1-\langle t\rangle s(z_k)]\exp[-\langle t\rangle s(z_k)L]\sum_{h=0}^{L-1}\frac{[\langle t\rangle s(z_k)L]^h}{h!}+\exp[-\langle t\rangle s(z_k)L]\frac{[\langle t\rangle s(z_k)L]^L}{L!}. \tag{48}$$

Since $(\langle t\rangle s)^L\exp(-\langle t\rangle sL)\le\exp(-L)$, the last term on the right-hand side of Eq. (48) is seen to vanish when L increases. Moreover, when the sum continues to ∞, the first term is to approach the value

$$\chi(L,z_k)\to m(z)\equiv 1-\langle t\rangle s(z). \tag{49}$$

Now that the backward probability $m(z)$ has been determined, we may investigate the periodicity of type II and its possibility of transition to type I. Serving as an example, the uniform distribution $p_{th}(t)$, i.e., $p_{th}(t)=1$ in $[0,1]$, is used here. We thus have $\langle t\rangle=1/2$ and $s(z)=e^z/(1+\kappa)$ where the result of Eq. (9) has been used with $y=0$. Letting $m=0$ in Eq. (49), i.e., a "superslip" event occurs in the system, we obtain

$$1=\frac{e^z}{2(1+\kappa)}. \tag{50}$$

For a given κ, the period of z is the solution of the above equation, i.e., $z_p=\ln 2(1+\kappa)$. Correspondingly, the period of T is simply that

$$T_p=N\int_0^{z_p}d\xi p(0,\xi)m(\xi). \tag{51}$$

From Eq. (50) it is clear that z_p is a monotonically increasing function of κ. As $z=1$ in Eq. (50) κ reaches its upper value, denoted by $\kappa_U=e/2-1\simeq 0.36$, beyond which the "superslip" event cannot occur any longer so that the system only shows the state of type I. Finally, we emphasize that z_p and T_p also may be determined for the general case of $p_{th}(t)$ by the above means.

3. Distributions $D(\Delta)$

Based on the forward and backward probabilities obtained in the preceding paragraphs, we can determine the distribution $D(\Delta)$ of shock with size Δ for the whole process when driving tip A steadily. It is derived next that the distribution $D(\Delta)$ is two distinct power laws whose exponents are -1.5 and -2.5 for type I and type II, respectively.

For type I, since the strain distribution $p(y,z)$ is of the stationary form, i.e., $p(y,z)=p_\infty(y)$, the backward probability given in Eq. (49) is essentially a constant $1-\alpha\langle t\rangle$. The distribution $D_I(\Delta)$ is essentially from the forward probability $\Psi(\Delta,z)$ of Eq. (31)

$$D_I(\Delta)=\Psi(\Delta,\infty)$$
$$=\frac{s(\infty)^{\Delta-1}}{(\Delta-1)!}I_0(s(\infty))\prod_{i=1}^{\Delta-1}\int_0^1 dx_i\gamma(x_i,0,\infty)$$
$$\times\phi_\Delta(s,x_i) \tag{52}$$

with

$$\phi_\Delta(s,x_i)=\exp\left[-s(\infty)\sum_{i=1}^{\Delta-1}x_i\right]\hat S_f\left[\sum_{i=1}^{\Delta-1}x_i\right]^{\Delta-1}. \tag{53}$$

Notice that $\gamma(x,0,\infty)=p_{th}(x)$. Since Eq. (52) is similar to the case of a model with prestrain distribution [40], we study the above formula in a similar way. Take $\hat P$, a permutation operator of the x_i's, to symmetrize the polynomial $\hat S_f(\sum_{i=1}^{\Delta-1}x_i)^{\Delta-1}$ in Eq. (53). Since the value of $D(\Delta)$ is invariant under any permutation of x_i's, and the total number of the permutations is $(\Delta-1)!$, we may replace the last term on the right-hand side of Eq. (53) with

$$\frac{1}{(\Delta-1)!}\sum_{\{\hat P\}}\hat P\hat S_f\left[\sum_{i=1}^{\Delta-1}x_i\right]^{\Delta-1},$$

where $\{\hat P\}$ indicates all the permutations. Using the mean-value theorem of integration we have

$$D_I(\Delta)=\frac{s(\infty)^{\Delta-1}}{(\Delta-1)!}I_0(s(\infty))\phi_\Delta(s,\theta_i) \tag{54}$$

with $0\le\theta_i\le 1$ for $i=1,2,\cdots,\Delta-1$. The θ_i's may take identical values, say θ, due to the invariance of the function ϕ_Δ under any permutations of x_i's. Considering there are a total number $(\Delta-1)^{\Delta-1}$ of terms in the polynomial $(\sum_{i=1}^{\Delta-1}x_i)^{\Delta-1}$ while only $\Delta^{\Delta-2}$ in $\hat S(\sum_{i=1}^{\Delta-1}x_i)^{\Delta-1}$ [36], we hence have

$$\phi_\Delta(s,\theta) = \Delta^{\Delta-2}\theta^{\Delta-1}\exp[-(\Delta-1)\theta s(\infty)].$$

Substituting the above equation into Eq. (54) and using the Stirling approximation $\Delta! = \Delta^\Delta e^{-\Delta}\sqrt{2\pi\Delta}$, we finally obtain

$$D_I(\Delta) = C_I \Delta^{-\frac{3}{2}} \exp(-\Delta/\Delta_0), \tag{55}$$

where $C_I = I_0(s(\infty))\exp[\theta s(\infty)]/[\sqrt{2\pi}\theta s(\infty)]$ and $\Delta_0 = 1/[\theta s(\infty) - 1 - \ln\theta s(\infty)]$. We have thus succeeded in showing that the exponent of the power law (55) is universal, independent of the threshold distribution $p_{th}(t)$. Simulations for type I also confirm the same power-law behavior in shock distributions as Eq. (55) (see Fig. 4). Given a κ, we increase N in the simulations and find that Δ_0 is seen to diverge.

Let us turn to the case for type II. The probability of a shock of size Δ starting at "time" z is given by the product of the forward and the backward probabilities Eqs. (31) and (44), i.e.,

$$W(\Delta,z) = \frac{s(z)^{\Delta-1}}{(\Delta-1)!} I_0^2(s(z)) \left[\prod_{i=1}^{\Delta-1}\int_0^1 dx_i \gamma(x_i,0,z)\right] \times \lim_{L\to\infty}\left[\prod_{i=2}^{L}\int_0^1 dx'_{k-i}\gamma(x'_{k-i},0,z)\right] \times\tilde{\phi}_\Delta(s,x_i,x'_{k-i}) \tag{56}$$

with

$$\tilde{\phi}_\Delta(s,x_i,x'_{k-i}) = \sum_{h=0}^{L-1}\frac{s(z)^h}{h!}\exp\left[-s(z)\sum_{i=1}^{\Delta-1}x_i\right] \times\exp\left[-s(z)\sum_{i=2}^{L}x'_{k-i}\right] \times\hat{S}_f\left[\sum_{i=1}^{\Delta-1}x_i\right]^{\Delta-1}\hat{S}_b\left[\sum_{i=2}^{L}x'_{k-i}\right]^h . \tag{57}$$

Also symmetrizing the polynomials on the right-hand side of Eq. (57) with all possible permutations of x's, we have

$$\tilde{\phi}_\Delta(s,x_i,x'_{k-i}) = \sum_{h=0}^{L-1}\frac{s(z)^h}{h!}\exp\left[-s(z)\sum_{i=1}^{\Delta-1}x_i\right]\exp\left[-s(z)\sum_{i=1}^{L}x'_{k-i}\right] \times\frac{1}{(\Delta+L-2)!}\sum_{\{\hat{P}\}}\hat{P}\hat{S}_f\left[\sum_{i=1}^{\Delta-1}x_i\right]^{\Delta-1}\hat{S}_b\left[\sum_{i=2}^{L}x'_{k-i}\right]^h . \tag{58}$$

We also can pick an identical value $\tilde{\theta}$ for all x's when the mean-value theorem of integration is applied to Eq. (56). Since there are a total number $(\Delta-1)^{\Delta-1}$ of terms in the polynomial $(\sum_{i=1}^{\Delta-1}x_i)^{\Delta-1}$ while only $\Delta^{\Delta-2}$ in $\hat{S}_f(\sum_{i=1}^{\Delta-1}x_i)^{\Delta-1}$, and $(L-1)^h$ in $[\sum_{i=2}^{L}x'_{k-i}]^h$ while only $(L-h)L^{h-1}$ in $\hat{S}_b[\sum_{i=2}^{L}x'_{k-i}]^h$ [36,40], we derive from Eq. (57) that

$$W(\Delta,z) = \frac{\Delta^{\Delta-1}}{\Delta!} I_0^2(s(z)) f^{\Delta-1}\exp[-(\Delta-1)f] \times\left\{\lim_{L\to\infty}\exp[-(L-1)f]\sum_{h=0}^{L-1}\frac{L-h}{h!L}(Lf)^h\right\}, \tag{59}$$

where $f(z) = \tilde{\theta}s(z)$ and $\tilde{\theta}\in[0,1]$. Dealing with the term inside the brace of the right-hand side of Eq. (59) in the same way as used in Eq. (47), we then get

$$W(\Delta,z) = \frac{\Delta^{\Delta-1}}{\Delta!}\frac{I_0^2(s(z))e^{2f}}{f}(1-f)(fe^{-f})^\Delta. \tag{60}$$

Finally, we must integrate Eq. (60) over z in a period, since a shock of size Δ may occur at any point of z before the superslip of the whole system. Thus the distribution of a shock of size Δ for type II is

$$D_{II}(\Delta) = \frac{\Delta^{\Delta-1}}{\Delta!\Gamma}\int_0^{z_p} dz p(0,z)\frac{I_0^2(s(z))e^{2f}}{f} \times(1-f)[fe^{-f}]^\Delta, \tag{61}$$

where $\Gamma = \int_0^{z_p} dz\ p(0,z)m(z)$, and z_p represents the pe-

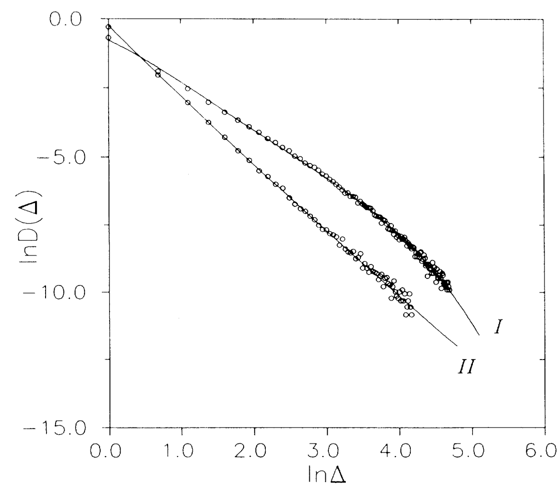

FIG. 4. The distributions $D(\Delta)$ of the shocks of size Δ as $N = 10\,000$, $\kappa = 0.25$, and $\lambda = 1$ for the two types: (I) type I; (II) type II. These could be well fitted as $D(\Delta)\sim\Delta^{-1.5}\exp(-\Delta/\Delta_0)$ for I and $D(\Delta)\sim\Delta^{-2.5}$ for II, respectively.

riod of type II such that $m(z_p) = 0$, i.e., $f(z_p) = 1$. We may employ a saddle-point approximation to perform the above integration. The function inside the bracket has a maximum of $1/e$ for $f = 1$ at the value z_p. Expanding around the maximum to second order in $z - z_p$, integrating, and using the Stirling approximation $\Delta! \simeq \Delta^\Delta e^{-\Delta}\sqrt{2\pi\Delta}$, we obtain the asymptotic result

$$D_{II}(\Delta) = C_{II}\Delta^{-\frac{5}{2}}, \tag{62}$$

with

$$C_{II} = \frac{p(0,z_p)e^2 I_0^2(s(z_p))}{\sqrt{2\pi}\tilde{\theta}s'_z(z_p)\Gamma}.$$

Thus we have succeeded in showing that the different power laws Eq. (55) and Eq. (62) for the states of the two types are all universal, independent of the threshold distributions. Note that these analytical results are obtained in the thermodynamic limit $N \to \infty$, and accord with the simulations shown in Fig. 4.

Based on the above discussions, it is clear that this system shows dynamical states of two types which are different in the stick-slip processes as illustrated in Fig. 2, in the evolutions of the strain distribution functions, and in the power laws of shock-size distributions as described in Eqs. (55) and (62). These two states could coexist over some region of the parameter space, e.g., at $\kappa = 0.25$ and $\lambda = 1.0$, depending on initial conditions. It is apparent that these two dynamical states in this model are not the same as claimed in Ref. [10].

D. The general λ case

Now that the property of the system for $\lambda = 1$ has been explored in the preceding paragraph, we continue to discuss the general case of λ. At first, we mention another special case of $\lambda = 0$ which is completely equivalent to that of a model with prestrain distribution [40]. In this special case, as discussed in Ref. [40], in spite of the shock size, the probability $p(y,z)$ of y_i's may be appropriately described by Eq. (8), reaching the stationary solution of Eq. (12). Correspondingly, $\gamma(x,y,z)$ is described as Eq. (15). Consequently, the distribution $D(\Delta)$ of a shock of size Δ is of a power law similar to Eq. (55) so that the system only evolves into a state of type I, independent of the initial conditions. We have confirmed these results by our simulations, and designate Fig. 5 as a representation.

The dynamical states of two types have been characterized in detail with the typical case of $\lambda = 1$. For an arbitrary $\lambda \in (0,1)$, there are also two states different in the stick-slip process, the strain evolutionary function, and the distribution of shock size. Depending on the initial distribution of strain and the parameters, the system may also evolve into one of the two states defined above. Because analyses for this case cannot be performed as simply as for $\lambda = 1$, we just employ simulations to discuss the different states of the system. With $p_{th}(t) = 1$ in $[0,1]$ and $N = 10\,000$, we show for the two types the stick-slip process of blocks and the distribution $D(\Delta)$ in Fig. 6 and Fig. 7, respectively, using representative values $\lambda = 0.8$ and $\kappa = 0.25$. For type I, the numerical results of $D(\Delta)$ may be well fitted as in Fig. 7 as $D(\Delta) \sim \Delta^{-\eta}\exp(-\Delta/\Delta_0)$ similar to Eq. (55) for the case $\lambda = 1$. In other words, we have $\eta \simeq 1.5$ and a finite characteristic size Δ_0 for general λ. For type II, the periodicity of the emergence of large-size shocks remains but the length of the period is smaller than for $\lambda = 1$. The distribution $D(\Delta)$ may be well fitted by the formula $D(\Delta) \sim \Delta^{-\eta}$, and otherwise the exponent η goes down beneath 2.5 (not smaller than 1.5) as λ decreases from 1.0 under a fixed κ. One may from the stick-slip process as in Fig. 6 still expect that the evolutionary functions of strain are stationary and periodical for type I and type II,

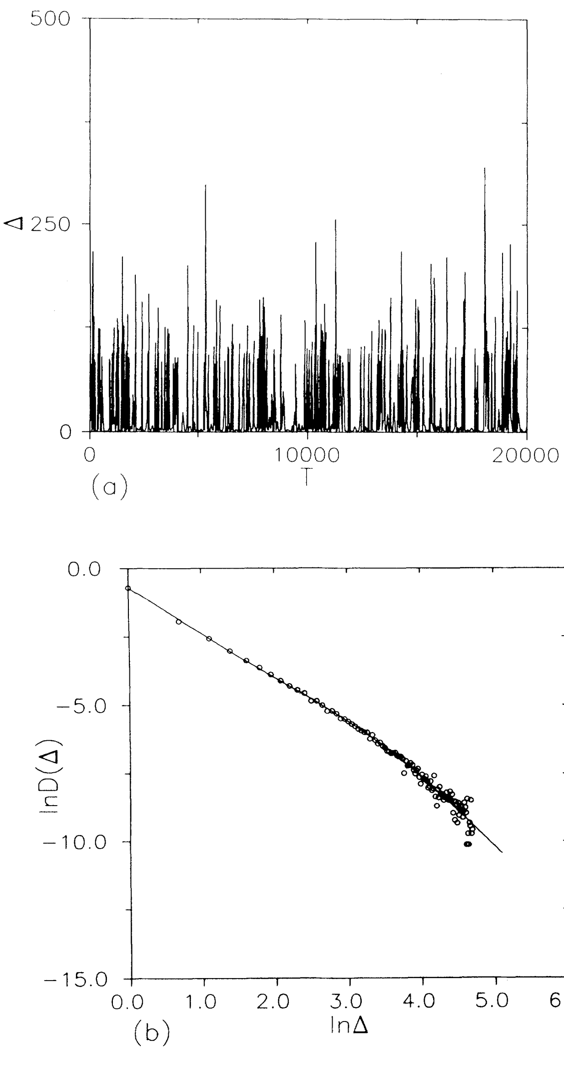

FIG. 5. The limiting case of $\lambda = 0$: (a) the stick-slip process; (b) the distribution $D(\Delta)$ of shock for size Δ, fitted well by $\Delta^{-1.5}\exp(-\Delta/\Delta_0)$. $N = 10\,000$ and $\kappa = 0.20$ are used here. There is only a state of type I in the system for this pair of parameter values.

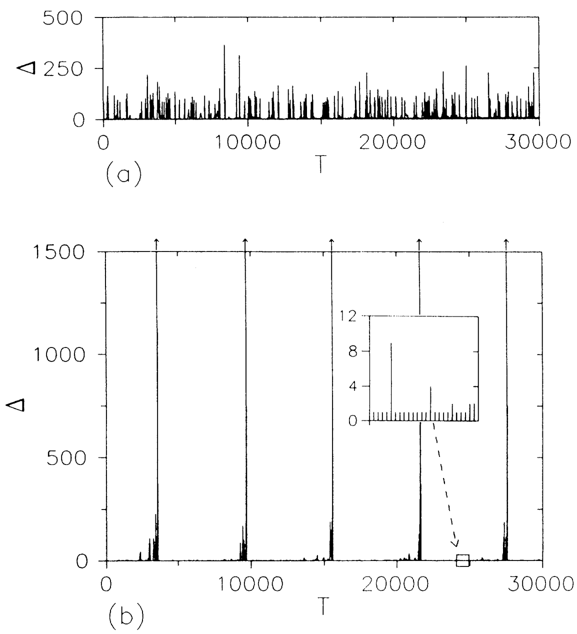

FIG. 6. The typical stick-slip processes for the states of two types. They are obtained with $N = 10\,000$, $\kappa = 0.25$, and $\lambda = 0.8$. The configurations of two types are different due to the initial states of the system. (a) type I; (b) type II. The lines with arrows away from the tops of the columns in (b) indicate that $\Delta \simeq 10\,000$.

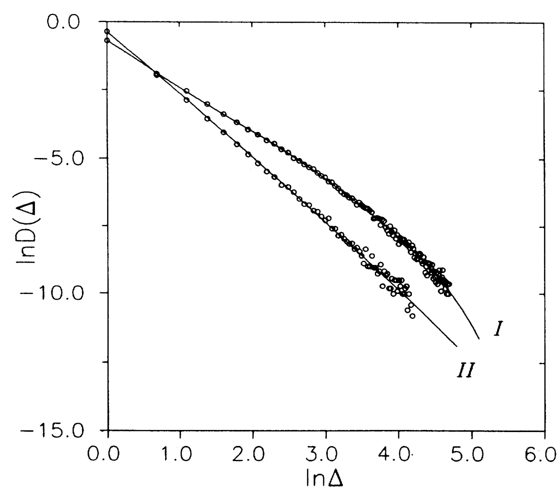

FIG. 7. The distributions $D(\Delta)$ of the shocks of size Δ as $N = 10\,000$, $\kappa = 0.25$, and $\lambda = 0.8$ for the two types: (I) type I; (II) type II. Those could be well fitted as $D(\Delta) \sim \Delta^{-1.5}\exp(-\Delta/\Delta_0)$ for I and $D(\Delta) \sim \Delta^{-2.2}$ for II, respectively.

respectively, in $\lambda \in (0,1)$. So one may understand that the system presented here displays two states which are distinct in three aspects as emphasized before.

IV. THE TRANSITION BETWEEN TWO STATES

We now come to investigate the dynamical transitions of the system by numerical simulations. By setting different initial conditions and surveying the stick-slip process, the exponent η, and the characteristic size Δ_0 for various values of κ and λ, we can get a phase diagram in the κ-λ plane (Fig. 8). A system with given values of parameters is represented by a point in the κ-λ plane. For a large N ($\geq 10\,000$), starting from a representative point in the coexistent region of parameter space, e.g., $(0.25, 1.0)$, to investigate the two possible states of the system, we move the representative point in three ways: (a) horizontally as $\kappa \to 0$ and λ unchanged; (b) vertically as $\lambda \to 0$ and κ unchanged; (c) along a curve $\lambda = 16\kappa^2$ toward the origin. In (a), the values of the exponent η for the two types do not change, while the period for type II decreases (as illustrated in Fig. 9), and type I abruptly disappears at $\kappa \simeq 0.15$. With $\kappa < 0.15$ any initial state leads to type II. In (b), as shown in Fig. 10, the exponent η for type II decreases continuously, and the state of type II disappears suddenly at $\lambda \simeq 0.5$. When $\lambda < 0.5$, the system always evolves into the state of type I. In (c), the states of two types coexist in the system while the characteristic size Δ_0 for type I increases, and the exponent η and period for type II decrease. As $\lambda \to 0$ and $\kappa \to 0$, the differences between these two states seem to disappear. So the system evolves into a self-organized critical state with the power-law behavior of shocks yielding $D(\Delta) \sim \Delta^{-1.5}$, and with a stationary strain distribution, independent of initial conditions [4]. It remains that the bounds of the phase diagram of

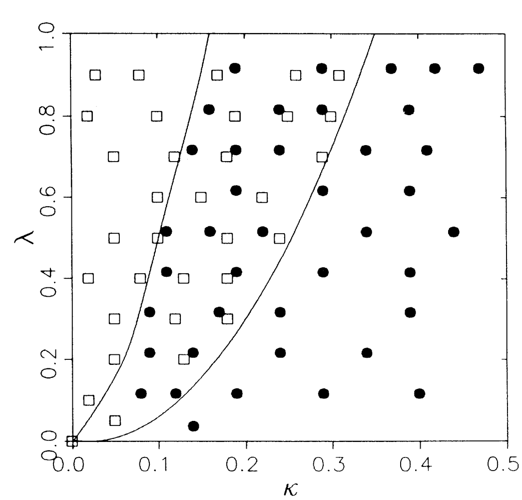

FIG. 8. The sketch of the phase diagram for the model as $N = 10\,000$. Symbol • indicates the existence of type I, and □ type II. The estimated boundaries roughly define the regions in which the corresponding state could exist.

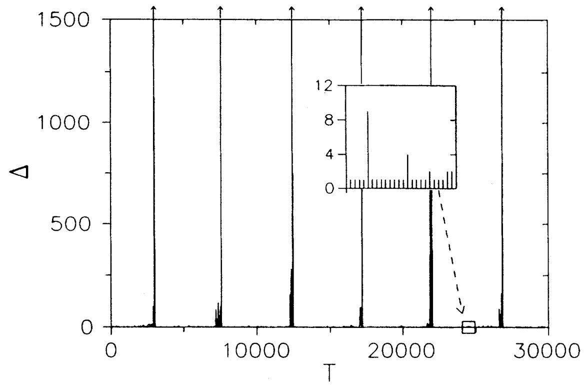

FIG. 9. The decrease of the period for type II as $\kappa \to 0$. As an example, with $N = 10\,000$ and $\lambda = 0.8$ fixed, but κ changed to 0.05, the period of the above stick-slip processes for type II has obviously decreased as compared with $\kappa = 0.25$ in Fig. 6(b).

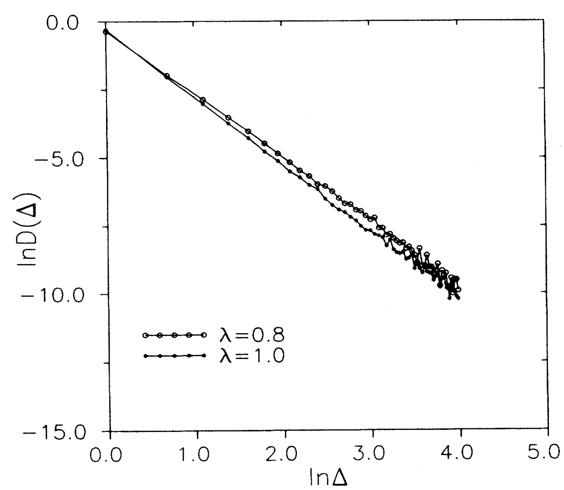

FIG. 10. The variance of the power-law exponent for type II as $\lambda \to 0$ under fixed $\kappa = 0.25$ in the κ-λ plane.

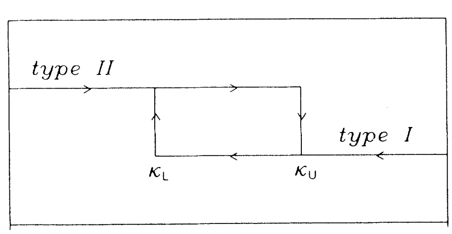

FIG. 11. An example of the hysteresis phenomenon of the model with $\lambda = 0.8$, $N = 10\,000$, and the change of κ per step equal to 0.05. κ_L and κ_U, respectively, measured to be about 0.15 and 0.30, are transition points of the type when κ is quasistatically changed along the opposite directions indicated with the arrows.

Fig. 8 slightly depend on N. From this phase diagram we can easily understand that hysteresis could exist in this model. For example, with a given $\lambda = 0.8$ and a uniform distribution $p_{th}(t)$, when $\kappa \in [0, 0.5]$ is quasistatically increased or decreased alternatively, one could find the phenomenon of hysteresis (Fig. 11). A continuous transition was previously observed in a self-organized critical cellular-automaton model equivalent to a driven spring-block model when the level of conservation varies [10]. However, the dynamical phase transition presented here is a different one since the transition takes place abruptly as some parameter (e.g., κ) changes.

V. CONCLUSIONS

The scaling properties of earthquakes has captured the imagination of many physicists, and led to an explosion of activity and publications. Various versions of the spring-block model of earthquakes have been proposed to understand such behaviors, and self-organized criticality (SOC) is perhaps the underlying mechanism [18,19,22]. As nonconservative ones, the models of earthquake robustly display SOC without the conservation law necessary in noisy driven diffusion systems [5].

In this paper, we investigate SOC in earthquakes through a modified Burridge-Knopoff model where there are two types of dynamical states which are different from each other in the stick-slip process of the blocks, the evolutionary distribution of strain, and the power-law behavior of shock distribution. For the limiting case $\lambda = 1$ we analytically obtain the strain distribution functions and the power laws for the two types. Simulations show that the two distinct states may coexist in some region of parameter space. A transition and a hysteresis phenomenon are seen when the parameters are appropriately changed. The phenomenon of hysteresis is perhaps the most interesting feature of our model.

Besides the earthquakes, there appear to be some other phenomena in nature related to the theoretical spring-block model of this paper. In a type-II superconductor, under small perturbations in the driving force, a system of pinned flux lattices shows an analogous stick-slip process where free-flow and stick regimes are found corresponding to large and small driving force, respectively, and it exhibits a power-law distribution characteristic of SOC [42]. In biology, the electrical activity of the pancreatic β cells in an intact Islet of Langerhans shows that a transition between active phase (chaotic spiking) and silent phase (regular bursting) takes place as the size of the clusters or the conductance of the cell is varied, and that two phases may coexist in some range of the conductance [43,44]. Also, it was reported that the probability density function of the dwell times in the open and closed conformational states displays scaling functions in protein dynamics [45]. There appears to be some underlying relationship between these natural phenomena and the theoretical model of this paper.

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